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p-Minors of a Doubly Stochastic Matrix at Which the Permanent Achieves a Minimum*†

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ABSTRACT

The author proves that if A is a matrix at which the permanent achieves a local minimum relative to the set of $n\times n$ doubly stochastic matrices, then for $a_{ij}=0$,

$$\operatorname{per}A(i|j)\geqslant\operatorname{per}A.$$

This paper is motivated by a long-standing conjecture of B. L. van der Waerden which states that the permanent achieves a unique minimum on the set D_n of $n\times n$ doubly stochastic matrices at the matrix J_n whose entires are each $1/n$ [7].

The author proves that *if A is a matrix at which the permanent achieves a local minimum relative to the set D_n, then for $a_{ij}=0$,* $\operatorname{per}A(i|j)\geqslant\operatorname{per}A$.

The above result has been proven by David London in [1]. Independently of him, the present author proved the above conclusion under the hypothesis that A was a matrix at which the permanent achieves an absolute minimum relative to D_n. Since London's proof was based on the duality theorem from linear programming, the author felt that it would be of interest to have a combinatorial proof. Indeed, the above result is of interest because it validates a proof by H Minc in [5, pp. 261–262].

A nonnegative matrix in which each row and column has sum 1 is said to be doubly stochastic. An $n\times n$ matrix A is said to have doubly stochastic pattern if there is an $n\times n$ doubly stochastic matrix B such that $a_{ij}=0$ if and only if $b_{ij}=0$. A square matrix A is said to be partly decomposable if there is an $s\times t$ zero submatrix such that $s+t=n$. A square matrix A is said to be fully indecomposable if it is not partly decomposable.

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Let A be an $n \times n$ matrix. Let per$A$ denote the permanent of A, a number obtained by taking the product of the entries in each diagonal of A and then summing over all of these products. The quantity per$A(i|j)$ is called the p-minor of a_{ij}. It is the permanent of the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the ith row and jth column.

We shall say that two nonzero entries $a_{i(1),j(1)}$ and $a_{i(k),j(k)}$ are chainable if there is a sequence $a_{i(1),j(1)},\ldots,a_{i(k),j(k)}$ of different, nonzero entries such that for $1 \leqslant r \leqslant k$, $i_r = i_{r+1}$ or $j_r = j_{r+1}$; and if $1 < h+1 < p \leqslant k$, then $i_h \neq i_p$ and $j_h \neq j_p$. We shall say that the matrix A is chainable if each nonzero entry can be chained to any other. The concept can be visualized by the movements of a rook with stationary positions on the nonzero entries of A. Observe that once the rook moves out of a row (column) it cannot return to that row (column).

We shall make use of the following results.

THEOREM A. *If A is a matrix at which the permanent achieves a local minimum relative to D_n, then A is fully indecomposable.*

The proof of Theorem A is identical to the one in [2, pp. 63–64].

THEOREM B. *If A is a matrix at which the permanent achieves a local minimum relative to the set D_n, then*

$$\operatorname{per}A(i|j) = \operatorname{per}A \quad \textit{if} \quad a_{ij} > 0.$$

The proof of Theorem B is an obvious modification of the one in [2, pp. 64–66].

The following theorem is proved in [6, pp. 68–69].

THEOREM C. *A square matrix A is fully indecomposable if and only if A has doubly stochastic pattern and is chainable.*

We shall now prove our main result.

THEOREM. *Let A be a matrix at which the permanent achieves a local minimum relative to D_n. Then for $a_{ij} = 0$,*

$$\operatorname{per}A(i|j) \geqslant \operatorname{per}A.$$

Proof. Let $a_{ij} = 0$. Since A is doubly stochastic, there are nonzero entries a_{ip} and a_{qj}. Since A is fully indecomposable by Theorem A, it follows by Theorem C that A is chainable. Thus there is a chain $a_{qj},\ldots,a_{ip}$. We can assume that the chain has the form $a_{qj}, a_{q,j(1)}, \ldots, a_{i(k),p}, a_{ip}$. If it did not, we would only need to change the appropriate end points and we would still have a chain which connected the ith row and jth column. Consider the matrix $A(f)$ which is obtained from A by placing f in (i,j), $a_{qj}-f$ in (q,j), $a_{q,j(1)}+f$ in $(q,j(1)),\ldots,$ $a_{i(k),p}+f$ in $(i(k),p)$, and $a_{ip}-f$ in (i,p), and leaving all other entries the same. Since $A(f)$ is doubly stochastic for $0 \leqslant f \leqslant \min\{a_{qj},\ldots,a_{ip}\}$, $\operatorname{per} A(f)$ is nondecreasing on some interval $[0,c]$. Hence $d(\operatorname{per} A(0))/df$ is nonnegative. By Theorem B, it follows that for any a_{rs} in the chain, $\operatorname{per} A(r|s) = \operatorname{per} A$. Hence

$$\operatorname{per} A(i|j) - \operatorname{per} A = \operatorname{per}(i|j) - \operatorname{per} A(i|p) = \frac{d \operatorname{per} A(0)}{df} \geqslant 0.$$

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